Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (Month 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
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Fuzzy Hilbert adjoint operator and its properties

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Received 27 March 2018; Revised 19 April 2018; Accepted 29 April 2018

Abstract. In this paper, definition of fuzzy real Hilbert adjoint operator is given and some of its properties are studied. We also prove existence theorem for fuzzy real Hilbert adjoint operator.

2010 AMS Classification: 46S40, 03E72

Keywords: Fuzzy Hilbert adjoint operator, Fuzzy self-adjoint operator.

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1. Introduction

Metric, norm and inner product structures are the main tools in functional analysis. So fuzzy metric, fuzzy norm and fuzzy inner product play the crucial role to develop fuzzy functional analysis. Several authors studied fuzzy metric space as well as fuzzy normed linear space and a large number of papers have been published. We refer some of them which are related to our work (please see [1, 2, 3, 6]

Study on fuzzy inner product spaces are relatively recent. Many authors introduced the concept of fuzzy inner product in different approaches (for reference please see [4, 5, 7, 8, 9, 11, 12, 14, 15].

In this paper, following the definition of fuzzy inner product given by Hasankhani et.al [9], an idea of fuzzy Hilbert adjoint operator is introduced and existence theorem for such operator is established.

It is to be noted that Hasankhani et.al [9] considered the fuzzy real number in the sense of Kaleva et. al [10] to define fuzzy inner product whose induced fuzzy norm is Felbin's type [6] fuzzy norm. In this paper we consider Xiao and Zhu [16] type fuzzy real number and the induced fuzzy norm is Bag and Samanta [3] type fuzzy norm. In [3], it is shown that all the result which are valid in Felbin's fuzzy norm [6] are also valid in Bag and Samanta [3] type fuzzy norm.

The organization of the paper is as follows:
Section 2, provides some preliminary results which are used in this paper. Definition of fuzzy Hilbert adjoint operator is given and its existence theorem is established in

Section 3. In Section 4, some properties of fuzzy Hilbert adjoint operator and fuzzy Hilbert self-adjoint operator are studied.

2. Preliminaries

In this section, some definitions and preliminary results are given which will be used in this paper.

According to Mizumoto and Tanaka [13], a fuzzy real number is a mapping

$$x : \ R \rightarrow [0 \ , \ 1]$$

over the set R of all reals.

x is called convex, if $x(t) \ \geq \ min \ (x(s) \ , \ x(r))$, where $s \leq \ t \ \leq r$.

If there exists $t_0 \in R$ such that $x(t_0) \ = 1$, then x is called normal.

For $0 < \alpha \leq 1$, α-level set of an upper semicontinuous convex normal fuzzy set of R (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha \ , \ b_\alpha]$, where $a_\alpha \ = -\infty$ and $b_\alpha \ = +\infty$ are admissible. When $a_\alpha \ = -\infty$, for instance, then $[a_\alpha \ , \ b_\alpha]$ means the interval $(-\infty \ , \ b_\alpha]$. Similar is the case when $b_\alpha = +\infty$.

x is called non-negative, if $x(t) \ = 0, \ \forall t \ < \ 0$.

For any real number r, $\bar{r}$ is defined by $\bar{r}(t) \ = 1$ if $t = \ r$ and $\bar{r}(t) \ = 0$ if $t \neq \ r$. Kaleva and Seikkala [10] (Felbin [6]) denoted the set of all convex, normal, upper semicontinuous fuzzy real numbers by E (R(I)) and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $G(R^*(I))$. A partial ordering " $\preceq$ " in E is defined by $\eta \ \preceq \ \delta$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq \ b^2_\alpha$, for all $\alpha \ \in \ (0 \ , \ 1]$, where $[\eta]_\alpha \ = \ [a^1_\alpha \ , \ b^1_\alpha]$ and $[\delta]_\alpha \ = \ [a^2_\alpha \ , \ b^2_\alpha]$. The strict inequality in E is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$, for each $\alpha \ \in (0 \ , \ 1]$.

According to Mizumoto and Tanaka [13], the arithmetic operations $\oplus$, $\ominus$, $\odot$, $\oslash$ on $E \times E$ are defined by:

$(x \oplus y)(t) \ = \ Sup_{s \in R} \ \ min \ \{x(s) \ , \ y(t-s)\}, \ \ t \in R,$
$(x \ominus y)(t) \ = \ Sup_{s \in R} \ \ min \ \{x(s) \ , \ y(s-t)\}, \ \ t \in R,$
$(x \odot y)(t) \ = \ Sup_{s \in R, s \neq 0} \ \ min \ \{x(s) \ , \ y(\frac{t}{s})\}, \ \ t \in R,$
$(\eta \oslash \delta)(t) \ = \ Sup_{s \in R} \ \ min \ \{\eta(st) \ , \ \delta(s)\}, \ \ t \in R.$

Definition 2.1 ([6])**.** The absolute value $|\eta|$ of $\eta \ \in F(R)$ is defined by:

$$|\eta|(t) = \begin{cases} max(\eta(\text{t}), \eta(\text{-t})) & \text{if } t \ \geq 0 \\ 0 & \text{if } t \ < 0. \end{cases}$$

Lemma 2.2 ([10])**.** *Let* $\eta, \gamma \ \in F(R)$ *and* $[\eta]_\alpha \ = [\eta^-_\alpha, \ \eta^+_\alpha], \ [\gamma]_\alpha \ = [\gamma^-_\alpha, \ \gamma^+_\alpha] \ \forall \ \alpha \in (\ 0, \ 1]$. *Then*

(1) $[\eta \oplus \gamma]_\alpha \ = [\eta^-_\alpha + \gamma^-_\alpha, \ \eta^+_\alpha + \gamma^+_\alpha]$,
(2) $[\eta \ominus \gamma]_\alpha \ = [\eta^-_\alpha - \gamma^+_\alpha, \ \eta^+_\alpha - \gamma^-_\alpha]$,
(3) $[\eta \odot \gamma]_\alpha \ = [\eta^-_\alpha \gamma^-_\alpha, \ \eta^+_\alpha \gamma^+_\alpha]$, *for* $\eta, \gamma \ \in F^+(R)$,
(4) $[\bar{1} \oslash \eta]_\alpha \ = \ [\frac{1}{\eta^+_\alpha}, \ \frac{1}{\eta^-_\alpha}]$, *if* $\eta^-_\alpha > 0$,
(5) $[|\eta|]_\alpha \ = [max(\ 0, \ \eta^1_\alpha, \ -\eta^2_\alpha), max(\ |\eta^1_\alpha|, \ \ |\eta^2_\alpha|)]$.

Definition 2.3 ([6])**.** Let X be a vector space over R. Let $|| \ ||\ : X \rightarrow R^*(I)$ and the mappings $L, U \ : [0 \ , \ 1] \times [0 \ , \ 1] \rightarrow [0 \ , \ 1]$ be symmetric, nondecreasing in both arguments and satisfying $L(0 \ , \ 0) = \ 0$ and $U(1 \ , \ 1) = \ 1$.

Write $[||x||]_\alpha = [||x||_\alpha^1 , ||x||_\alpha^2]$ for $x \in X,\ 0 < \alpha \leq 1$ and suppose for all $x \in X$, $x \neq \underline{0}$, there exists $\alpha_0 \in (0 , 1]$ independent of x such that for all $\alpha \leq \alpha_0$,

(A) $||x||_\alpha^2 < \infty$,

(B) $\inf||x||_\alpha^1 > 0$.

The quadruple $(X , || ||, L , U)$ is called a fuzzy normed linear space and $|| ||$ is a fuzzy norm, if

(i) $||x|| = \bar{0}$ if and only if $x = \underline{0}$ (the null vector),

(ii)$||rx|| = |r|||x||,\ x \in X,\ r \in R$,

(iii) for all $x, y \in X$,

(a) whenever $s \leq ||x||_1^1,\ t \leq ||y||_1^1,\ s + t \leq ||x + y||_1^1$ and $||x + y||(s + t) \geq L(||x||(s) , ||y||(t))$.

(b) whenever $s \geq ||x||_1^1,\ t \geq ||y||_1^1,\ s + t \geq ||x + y||_1^1$ and $||x + y||(s + t) \leq U(||x||(s) , ||y||(t))$.

Note 2.4 ([6])**.** For the case when $U = \bigvee(max)$ and $L = \bigwedge(min)$, then the condition (iii) is equivalent to $||x + y|| \preceq ||x|| \oplus ||y||$ and $|| ||_\alpha^i :\ i = 1, 2$ are crisp norms on X and $(X , || ||, L , U)$ is simply denoted as $(X , || ||)$.

Definition 2.5 ([16])**.** A mapping $\eta : R \to [0,\ 1]$ is called a fuzzy real number, whose α level set is denoted by $[\eta]_\alpha = \{t\ :\ \eta(t) \geq \alpha\},\ 0 < \alpha \leq 1$, if it satisfies two axioms:

(N1) there exists $t_0 \in R$ such that $\eta(t_0) = 1$,

(N2) for each $\alpha \in (0,\ 1]$; $[\eta]_\alpha = [\eta_\alpha^-,\ \eta_\alpha^+]$, where $-\infty,\ \eta_\alpha \leq +\infty$.

The set of all fuzzy real numbers is denoted by F.

Since to each $r \in R$, one can consider $r \in F$ defined by $r(t) = 1$ if $t = r$ and $r(t) = 0$ if $t \neq r$, R can be embedded in F.

Lemma 2.6 ([16])**.** *$\eta \in F$ if and only if $\eta : R \to [0,\ 1]$ satisfies:*

(1) η normal, convex and upper semicontinuous,

(2) $lim_{t\to\infty}\eta(t) = 0$.

Definition 2.7 ([16])**.** Let $\eta \in F$. Then η is called a positive fuzzy real number, if $\eta(t) = 0\ \forall t < 0$. The set of all positive fuzzy real numbers is denoted by F^+.

Definition 2.8 ([3])**.** Let X be a linear space over R. Let $|| || : X \to F^+$ be a mapping satisfying:

(i) $||x|| = \bar{0}$ if and only if $x = \underline{0}$ (the null vector),

(ii)$||rx|| = |r|||x||,\ x \in X,\ r \in R$,

(iii) for all $x,\ y \in X$, $||x + y|| \preceq ||x|| \oplus ||y||$ and

(A'): $x \neq 0 \Rightarrow ||x||(t) = 0,\ \forall t \leq 0$.

Then $(X,\ || ||)$ is called a fuzzy normed linear space and $|| ||$ is called a fuzzy norm on X.

Corollary 2.9 ([2])**.** *Let $(X,\ || ||)$ be a fuzzy normed linear space. If for $x \in X$,*

$$[||x||]_\alpha = [||x||_\alpha^1,\ ||x||_\alpha^2],\ 0 < \alpha \leq 1,$$

then

(1) $||x||_\alpha^1 = Sup_{\beta<\alpha}||x||_\beta^1$,

(2) $||x||_\alpha^2 = Inf_{\beta<\alpha}||x||_\beta^2$.

Definition 2.10 ([9])**.** Let X be a vector space over R. A fuzzy inner product on X is a mapping $< ., \ . >: X \times X \to F(R)$ (set of fuzzy real numbers) such that for all vectors $x, \ y, \ z \in X$ and all $r \in R$,

(IP1) $\langle x+y, \ z\rangle = \langle x, \ z\rangle \oplus \langle y, \ z\rangle$,

(IP2) $\langle rx, \ y\rangle = \bar{r} \odot \langle x, \ y\rangle$,

(IP3) $\langle x, \ y\rangle = \langle y, \ x\rangle$,

(IP4) $\langle x, \ x\rangle \succeq \bar{0}$

(IP5) $\inf\limits_{\alpha\in(0 \ , \ 1]} \langle x, \ x\rangle_{\alpha}^{-} > 0$, if $x \neq \underline{0}$,

(IP6) $\langle x, \ x\rangle = \bar{0}$ if and only if $x = \underline{0}$.

The vector space X equipped with a fuzzy inner product is called a fuzzy inner product space.

A fuzzy inner product on X defines a fuzzy number

$$||x|| = \sqrt{\langle x, \ x\rangle}, \qquad \forall x \in X.$$

This is a well defined fuzzy norm.
A fuzzy Hilbert space is a complete fuzzy inner product space.

Theorem 2.11 ([7])**.** *Let X be a vector space over R and $< ., \ . >: X \times X \to F(R)$ be a fuzzy inner product (Hasankhani type). Let $[< x, \ y >]_{\alpha} = [< x, \ y >_{\alpha}^{1}, < x, \ y >_{\alpha}^{2}] \forall \ \alpha \in (\ 0, \ 1]$. Then $\{< ., \ . >_{\alpha}^{1}: \ \alpha \in (\ 0, \ 1]\}$ and $\{< ., \ . >_{\alpha}^{2}: \ \alpha \in (\ 0, \ 1]\}$ are families of crisp inner products from $X \times X \to R$.*

Definition 2.12 ([3])**.** Let $(X, || \ ||)$ and $(Y, || \ ||^{*})$ be two fuzzy normed linear spaces and $T : X \to Y$ be a linear operator. T is said to be strongly fuzzy bounded, if there exists a real number $k > 0$ such that $||Tx||^{*} \oslash ||x|| \ \preceq \ \bar{k} \ \forall x (\neq \underline{0}) \in \ X$.

Note 2.13. In this paper, we use fuzzy bounded operator T instead of strongly fuzzy bounded.

Proposition 2.14 ([3])**.** *Let $T : (X, || \ ||_{1}) \to (Y, || \ ||_{2})$ be a strongly fuzzy bounded linear operator and $\{[||T||_{\alpha}^{*1}, ||T||_{\alpha}^{*2}]; \alpha \in (0, \ 1]\}$ be a family of nested bounded closed intervals of real numbers. Define a function $||T||^{*} : R \to [0, \ 1]$ by:*

$$||T||^{*}(t) = \vee\{\alpha \ \in \ (0, \ 1] : t \in [||T||_{\alpha}^{*1}, ||T||_{\alpha}^{*2}]\}.$$

Then $||T||^{*}$ is a fuzzy real number (fuzzy interval) and it is the fuzzy norm of T.

Theorem 2.15 ([3])**.** *The set $B(X, Y)$ of all strongly fuzzy bounded linear operators from a fuzzy normed linear space $(X, || \ ||)$ to a fuzzy normed linear space $(Y \ || \ ||)$ is a linear space with respect to usual linear operations.*

Definition 2.16 ([7])**.** Let X and Y be vector spaces over the field R. Then a fuzzy sesquilinear form h on $X \times Y$ is a mapping $h : X \times Y \to F(R)$ such that for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$ and all scalars $\alpha, \ \beta$ the following conditions hold:

(i) $h(x_1 + x_2, \ y) = h(x_1, \ y) \oplus h(x_2, \ y)$,

(ii) $h(x, \ y_1 + y_2) = h(x, \ y_1) \oplus h(x, \ y_2)$,

(iii) $h(\alpha x, \ y) = \bar{\alpha} \odot h(x, \ y)$,

(iv) $h(x, \ \beta y) = \bar{\beta} \odot h(x, \ y)$.

$\bar{\alpha}$ denotes the fuzzy real number corresponding to α.

Definition 2.17 ([7]). Let h be a fuzzy sesquilinear form on $X \times Y$, where X and Y are real fuzzy normed linear spaces. h is said to be bounded, if $\exists$ a real number k such that

$$|h(x,\ y)| \oslash (||x|| \odot ||y||) \preceq \bar{k},\ \forall\ (x, y) \in X \times Y - \{(0, 0)\}.$$

Here $[|h(x,\ y)|]_\alpha = [\ max\ \{0, h^1_\alpha(x,\ y), -h^2_\alpha(x,\ y)\},\ max\ \{|h^1_\alpha(x,\ y)|, |h^2_\alpha(x,\ y)|\}]$, $\forall\ \alpha \in (0,\ 1]$.

Let h be a bounded sesquilinear form on $X\ \times\ Y$. Then $\exists\ k \in R$ such that

$$|h(x,\ y)| \oslash (||x|| \odot ||y||) \preceq \bar{k},\ \forall\ (x, y) \in X \times Y - \{(0, 0)\}.$$

Let $A = max\ \{0, h^1_\alpha(x, y), -h^2_\alpha(x, y)\}$ and $B = max\ \{|h^1_\alpha(x, y)|, |h^2_\alpha(x,\ y)|\}$. Then

$$\frac{A}{||x||^2_\alpha ||y||^2_\alpha} \leq k \text{ and } \frac{B}{||x||^1_\alpha ||y||^1_\alpha} \leq k,\ \forall\ \alpha \in (0,\ 1].$$

Define $||h||^{*1}_\alpha = \bigvee_{(x,\ y) \in X \times Y - \{(0,0)\}} \quad \frac{A}{||x||^2_\alpha ||y||^2_\alpha}$
and

$$||h||^{*2}_\alpha = \bigvee_{(x,\ y) \in X \times Y - \{(0,0)\}} \quad \frac{B}{||x||^1_\alpha ||y||^1_\alpha}.$$

Note 2.18 ([7]). $\{||h||^{*1}_\alpha; \alpha \in (0, 1]\}$ is an ascending family of norms and $\{||h||^{*2}_\alpha : \alpha \in (0, 1]\}$ is a descending family of norms and moreover $\{[||h||^{*1}_\alpha, ||h||^{*2}_\alpha] : \alpha \in (0, 1]\}$ is a family of nested bounded closed intervals of real numbers.

If a function $||h||^* : R \to [0, 1]$ given by:

$$||h||^*(t) = \vee\{\alpha \in (0, 1] : t \in [||h||^{*1}_\alpha, ||h||^{*2}_\alpha]\},$$

then $||h||^*$ is fuzzy norm of h.

Theorem 2.19 ([7], Riesz). *Let H_1, H_2 be two fuzzy Hilbert spaces and $h : H_1 \times H_2 \to F(R)$ be a bounded fuzzy sesquilinear form. Assume that for $h \neq \bar{o}$, $h^1_\alpha(x, y).h^2_\alpha(x, y) > 0$ and $\{y \in\ H_2; h(x, y) = 0,\ \ \forall x \in H_1\}$ is complete w.r.t. $||\ ||^1_\alpha\ \ \forall\alpha \in (0, 1]$, where $[||y||]_\alpha = [||y||^1_\alpha,\ ||y||^2_\alpha]$, and $||\ ||$ is the induced fuzzy norm by the fuzzy inner product of H_2. Then h can be represented as $h(x, y) =< Sx, y >$ where $S : H_1 \to H_2$ is a bounded linear operator and uniquely determined by h and has the norm $||h|| = ||S||$.*

3. Fuzzy Hilbert adjoint operator

As we consider fuzzy real numbers and H is a linear space over R, so H is a real Hilbert space. Throughout this paper Hilbert spaces are taken as real Hilbert spaces.

In this section fuzzy Hilbert adjoint operator is defined and thereafter its existence theorem is established.

Definition 3.1. Let $T : H_1 \to H_2$ be a fuzzy bounded linear operator where H_1 and H_2 are fuzzy Hilbert spaces. Then the fuzzy Hilbert adjoint operator T^* of T is the operator $T^* : H_2 \to H_1$ such that

$$< Tx, y >=< x, T^*y >,\ \forall x \in H_1 \textit{ and } y \in H_2.$$

Note 3.2. Let $(H_2, <.,.>_0)$ be a Hilbert space.
Define

$$< x, y > (t) = \begin{cases} 1 & \text{if } t =< x, y >_0 \\ 0 & \text{if } t = 0. \end{cases}$$

Then $< .,. >$ is a fuzzy real number,
where $[< x, y >]_\alpha = [< x, y >_0, < x, y >_0]$, $\forall \alpha \in (0,1]$ and $< .,. >$ defines a complete fuzzy inner product on H_2

Now we prove the existence of Hilbert adjoint operator $T^* : H_2 \to H_1$ for a given fuzzy bounded linear operator $T : H_1 \to H_2$.

Theorem 3.3. *(Existence Theorem) Let $T : H_1 \to H_2$ be a strongly fuzzy bounded linear operator where H_1 is a fuzzy Hilbert space and H_2 is given in Note 3.2. Then fuzzy adjoint operator T^* of T exists uniquely and $||T|| = ||T^*||$, if we assume that for $h \neq \bar{o}$, $h^1_\alpha(y, x).h^2_\alpha(y, x) > 0$ and $\{y \in H_2; h(y, x) = \bar{0},\ \forall x \in H_1\}$ is complete w.r.t. $||\ ||^1_\alpha$ (left α-norm of H_2), $\forall \alpha \in (0,\ 1]$, where h is given by $h(y, x) =< y, Tx >$.*

Proof. Since the inner product is sesquilinear, $h(y, x) =< y, Tx >$ defines a fuzzy sesquilinear form on $H_2 \times H_1$ and T is linear, where H_1 and H_2 are two Hilbert Spaces. Also

$$\begin{aligned} h(y,\ \alpha x_1 + \beta x_2) &=< y,\ T(\alpha x_1 + \beta x_2) > \\ &=< y,\ \alpha T x_1 + \beta T x_2 > \text{[Since } T \text{ is linear]} \\ &= \bar{\alpha} \odot < y,\ Tx_1 > \ \oplus \bar{\beta} \odot < y,\ Tx_2 > \\ &= \bar{\alpha} \odot h(y,\ x_1)\ \oplus\ \bar{\beta} \odot h(y,\ x_2)\ \ \forall \alpha, \beta \in R. \end{aligned}$$

Then h is linear. By the Schwarz inequality,

$$|h(y, x)| = |< y, Tx >| \preceq ||y|| \odot ||Tx||$$

$$\Rightarrow |h(y, x)| \oslash (||x|| \odot ||y||) \preceq (||y|| \odot ||Tx||) \oslash (||x|| \odot ||y||), \text{ when } x,\ y \neq \theta$$

$$\Rightarrow \frac{|h(y,x)|^1_\alpha}{||x||^2_\alpha ||y||^2_\alpha} \leq \frac{||y||^1_\alpha}{||y||^2_\alpha} . \frac{||Tx||^1_\alpha}{||x||^2_\alpha} \leq 1.k = k,\ \forall \alpha \in (0,1]$$

[Since $||y||^1_\alpha = ||y||^2_\alpha$ in H_2 space and since T is bounded

$$||Tx|| \oslash ||x|| \preceq \bar{k},\ \text{for some} k > 0]$$

Thus $\frac{|h(y,x)|^1_\alpha}{||x||^2_\alpha ||y||^2_\alpha} \leq k, \quad \forall \alpha \in (0,1]$ and $x, y \neq \theta$.

Now $\frac{|h(y,x)|^2_\alpha}{||x||^1_\alpha ||y||^1_\alpha} \leq \frac{||y||^2_\alpha}{||y||^1_\alpha} . \frac{||Tx||^2_\alpha}{||x||^1_\alpha} \leq 1.k = k,\ \forall \alpha \in (0,1]$. So

$$\frac{|h(y,x)|}{||x||\ ||y||} \preceq \bar{k}, \quad \forall x \in H_1 - \{\theta\}, y \in H_2 - \{\theta\}.$$

Hence $h(y, x)$ is bounded.

Now by Riesz Theorem (Theorem 2.19), we have, $h(y, x) =< T^* y, x >$ (putting T^* in place of the linear operator S), where $T^* : H_2 \to H_1$ is a bounded linear operator and $||h||^* = ||T^*||$ and $h(y, x) = \bar{h}(y, x)$. On the other hand,

$$\frac{|h(y,x)|^1_\alpha}{||x||^2_\alpha ||y||^2_\alpha} \leq \frac{||Tx||^1_\alpha}{||x||^2_\alpha},\ \forall \alpha \in (0,1] \text{ and } x, y \neq \theta.$$

Taking supremum on both sides, we get,

$$\Rightarrow \bigvee_{(y,x) \in H_2 \times H_1 - \{(\theta,\theta)\}} \frac{|h(y,\ x)|^1_\alpha}{||x||^2_\alpha ||y||^2_\alpha} \leq \bigvee_{x \in H_1 - \{\theta\}} \frac{||Tx||^1_\alpha}{||x||^2_\alpha}$$

$$\Rightarrow ||h||^{*1}_\alpha \leq ||T||^1_\alpha. \tag{3.3.1}$$

Now $h(Tx,x) =< Tx, Tx >= ||Tx||^2 \Rightarrow h^1_\alpha(Tx,x) = (||Tx||^1_\alpha)^2, \ \ \forall\alpha \in (0,1]$.
Since $h^1_\alpha(Tx,x) = h^2_\alpha(Tx,x)$,

$$||Tx||^1_\alpha = ||Tx||^2_\alpha, \ \ \forall\alpha \in (0,1]$$
$$\Rightarrow (||Tx||^1_\alpha)^2 = h^1_\alpha(Tx,x) \leq ||h||^{*1}_\alpha ||Tx||^2_\alpha ||x||^2_\alpha = ||h||^{*1}_\alpha ||Tx||^1_\alpha ||x||^2_\alpha$$
$$\Rightarrow \frac{||Tx||^1_\alpha}{||x||^2_\alpha} \leq ||h||^{*1}_\alpha.$$

Taking supremum, we get,

$$\bigvee_{x\in H_1-\{\theta\}} \frac{||Tx||^1_\alpha}{||x||^2_\alpha} \leq ||h||^{*1}_\alpha$$
$$\Rightarrow ||T||^1_\alpha \leq ||h||^{*1}_\alpha$$
$$\Rightarrow ||h||^{*1}_\alpha \geq ||T||^1_\alpha. \tag{3.3.2}$$

Then from (3.3.1) and (3.3.2), we get,

$$||h||^{*1}_\alpha = ||T||^1_\alpha, \ \ \forall\alpha \in (0,1]. \tag{3.3.3}$$

Also

$$\begin{aligned} ||h||^{*2}_\alpha &= \bigvee_{x\in H_1-\{\theta\},\ y\in H_2-\{\theta\}} \frac{|<y,Tx>|^2_\alpha}{||y||^1_\alpha ||x||^1_\alpha} \\ &\geq \bigvee_{x\in H_1-\{\theta\},\ Tx\in H_2-\{\theta\}} \frac{|<Tx,\ Tx>|^2_\alpha}{||Tx||^1_\alpha ||x||^1_\alpha} \\ &= \bigvee_{x\in H_1-\{\theta\},\ Tx\in H_2-\{\theta\}} \frac{||Tx||^2_\alpha ||Tx||^2_\alpha}{||Tx||^1_\alpha ||x||^1_\alpha} \\ &\geq \bigvee_{x\in H_1-\{\theta\},\ y\in H_2-\{\theta\}} \frac{||Tx||^1_\alpha ||Tx||^2_\alpha}{||Tx||^1_\alpha ||x||^1_\alpha} = ||T||^2_\alpha. \end{aligned}$$

Thus $||h||^{*2}_\alpha \geq ||T||^2_\alpha, \ \ \forall\alpha \in (0,1]$. (3.3.4)

For $\beta < \alpha,\ \alpha,\beta \in (0,1]$, we have,

$$\begin{aligned} &||h||^{*2}_\alpha \leq \ ||h||^{*2}_\beta \\ &= \bigvee_{(y,x)\in H_2\times H_1-\{(\theta,\theta)\}} \frac{h^2_\beta(y,\ x)}{||x||^1_\beta ||y||^1_\beta} \\ &= \bigvee_{(y,x)\in H_2\times H_1-\{(\theta,\theta)\}} \frac{h^1_\beta(y,x)}{||x||^1_\beta ||y||^1_\beta} \quad \text{[Since } h(y,x) = \bar{h}(y,x)\text{]}. \\ &= \bigvee_{x\in H_1-\{\theta\},\ y\in H_2-\{\theta\}} \frac{<y,Tx>^1_\beta}{||x||^1_\beta ||y||^1_\beta} \\ &\leq \bigvee_{x\in H_1-\{\theta\},\ y\in H_2-\{\theta\}} \frac{||Tx||^1_\beta ||y||^1_\beta}{||x||^1_\beta ||y||^1_\beta} \\ &= \bigvee_{x\in H_1-\{\theta\}} \frac{||Tx||^1_\beta}{||x||^1_\beta} \\ &\leq \bigvee_{x\in H_1-\{\theta\}} \frac{||Tx||^2_\beta}{||x||^1_\beta} \\ &= ||T||^2_\beta. \end{aligned}$$

So $||h||^{*2}_\alpha \leq \ ||T||^2_\beta,\ \forall\beta < \alpha$.
By taking infimum, we have, $||h||^{*2}_\alpha \leq \bigwedge_{\beta<\ \alpha} ||T||^2_\beta$. From Corollary 2.9,

$$||h||^{*2}_\alpha \leq \ ||T||^2_\alpha, \ \ \forall\alpha \in (0,1]. \tag{3.3.5}$$

From (3.3.4) and (3.3.5),

$$||h||^{*2}_\alpha = ||T||^2_\alpha, \ \ \forall\alpha \in (0,1]. \tag{3.3.6}$$

Hence by (3.3.3) and (3.3.6), $||h||^* = ||T||$.
Also $||h||^* = ||T^*||$. Therefore $||T|| = ||T^*||$. □

4. Properties of fuzzy Hilbert adjoint operator

In this section some basic properties of fuzzy Hilbert adjoint operator are studied.

Theorem 4.1. *Let H_1, H_2 be two fuzzy Hilbert spaces and $S : H_1 \to H_2$ and $T : H_1 \to H_2$ be two fuzzy bounded linear operators and α any scalar.Then*
(1) $< T^* y, x >=< y, Tx >$,
(2) $(S+T)^* = S^* + T^*$,
(3)$(\alpha T)^* = \alpha T^*$(*$\alpha$ is a real scalar),*
(4) $T^{**} = T$,
(5) $(ST)^* = T^* S^*$ *(assuming $H_1 = H_2 = H$).*
If we consider the fuzzy Hilbert space H_2 as in Note 3.2 and $H_1 = H_2 = H$, then following results hold:
(6) $||TT^*||^* = ||T^*T||^* = (||T||^*)^2$,
(7) $T^*T = 0$ *iff* $T = 0$.

Proof. (1) It is obvious, since $< .,. >$ is symmetric in both the arguments.
(2) $$\begin{aligned} < x, (S+T)^* y > &=< (S+T)x, y >=< Sx + Tx, y > \\ &=< Sx, y > \oplus < Tx, y >=< x, S^* y > \oplus < x, T^* y > \\ &=< x, (S^* + T^*) y > . \end{aligned}$$
Then $< x, (S+T)^* y >^1_\alpha =< x, (S^* + T^*) y >^1_\alpha$. Thus

$$< x, \{(S+T)^* - (S^* + T^*)\} y >^1_\alpha = 0, \quad \forall x \in H_1 \text{ and } \forall y \in H_2.$$

Similarly, $< x, \{(S+T)^* - (S^* + T^*)\} y >^2_\alpha = 0$, So $(S+T)^* = S^* + T^*$, since $< \, , \, >^1_\alpha$ and $< \, , \, >^2_\alpha$ are crisp inner products $\forall \alpha \in (0,1]$
(3) $$\begin{aligned} < (\alpha T)^* y, x > &=< y, (\alpha T) x >=< y, \alpha(Tx) > \\ &= \bar{\alpha} \odot < y, Tx) >= \bar{\alpha} \odot < T^* y, x) > \\ &=< \alpha T^* y, x >, \quad \forall x \in H_1 \text{ and } \forall y \in H_2. \end{aligned}$$
Then $\forall \alpha \in (0,1]$, we have $< (\alpha T)^* y, x >^1_\alpha =< \alpha T^* y, x >^1_\alpha$. Thus

$$< \{(\alpha T) - \alpha T^*\} y, x >^1_\alpha = 0.$$

Similarly, $< \{(\alpha T) - \alpha T^*\} y, x >^2_\alpha = 0$. So $(\alpha T)^* = \alpha T^*$.
(4) $$\begin{aligned} < Tx, y > &=< x, T^* y > \text{ [From definition]} \\ &=< T^* y, x > \text{ [Since } < .,. > \text{ is symmetric]} \\ &=< y, T^{**} x > \\ &=< T^{**} x, y > \end{aligned}$$
Then $\forall \alpha \in (0, \ 1]$ and $\forall x \in X, \ y \in Y$,

$$< Tx, y >^1_\alpha =< T^{**} x, y >^1_\alpha \ \text{ and } \ < Tx, y >^2_\alpha =< T^{**} x, y >^2_\alpha .$$

Since $< .,. >^1_\alpha$ and $< .,. >^2_\alpha$ are crisp inner products,

$$< (T - T^{**}) x, y >^1_\alpha = 0, \quad < (T - T^{**}) x, y >^2_\alpha = 0, \ \forall \alpha \in (0,1].$$

Thus $T - T^{**} = 0$. So $T^{**} = T$.
(5) $$\begin{aligned} < x, (ST)^* y > &=< (ST) x, y >=< S(Tx), y > \\ &=< Tx, S^* y >=< x, T^*(S^* y) > \\ &=< x, (T^* S^*) y >. \end{aligned}$$
Then $\forall \alpha \in (0,1]$ and $\forall x, y \in H$, we have $< x, (ST)^* y >^1_\alpha =< x, (T^* S^*) y >^1_\alpha$.
Thus $< x, \{(ST) - (T^* S^*)\} y >^1_\alpha = 0$.
Similarly, $< x, \{(ST) - (T^* S^*)\} y >^2_\alpha = 0$. So $(ST)^* = T^* S^*$.

(6)By Schwarz inequality, we have,
$||Tx||^2 =< Tx, Tx >=< x, T^*Tx >$
$=< T^*Tx,\ x >$
$\preceq ||T^*Tx|| \odot ||x||$
$\preceq ||T^*T|| \odot ||x||^2.$
Then $||Tx||^2 \preceq ||T^*T|| \odot ||x||^2$. Thus $||Tx||^2 \oslash ||x||^2 \preceq (||T^*T|| \odot ||x||^2) \oslash ||x||^2$, since $||x|| \succ \bar{0}$, for $x \neq \theta$.

Now $\forall \alpha \in (0, 1]$ and $x \neq \theta$, since $||x||^1_\alpha = ||x||^2_\alpha$,

$$(\frac{||Tx||^1_\alpha}{||x||^2_\alpha})^2 \leq ||T^*T||^1_\alpha (\frac{||x||^1_\alpha}{||x||^2_\alpha})^2 \leq ||T^*T||^1_\alpha .1.$$

Taking sup on $x \in H - \{\theta\}$, we have,

$$(\bigvee_{x \in H - \{\theta\}} \frac{||Tx||^1_\alpha}{||x||^2_\alpha})^2 \leq ||T^*T||^1_\alpha.$$

From Theorem 3.3,

$$(||T||^1_\alpha)^2 \leq ||T^*T||^1_\alpha \leq ||T^*||^1_\alpha ||T||^1_\alpha = ||T||^1_\alpha ||T||^1_\alpha.$$

So $(||T||^1_\alpha)^2 = ||T^*T||^1_\alpha$. (4.1.1)

Also $(\frac{||Tx||^2_\alpha}{||x||^1_\alpha})^2 \leq ||T^*T||^2_\alpha (\frac{||x||^2_\alpha}{||x||^1_\alpha})^2 = ||T^*T||^2_\alpha .1$
Taking sup, we get, $(\bigvee_{x \in H - \{\theta\}} \frac{||Tx||^2_\alpha}{||x||^1_\alpha})^2 \leq ||T^*T||^2_\alpha$. From Theorem 3.3,

$$(||T||^2_\alpha)^2 \leq ||T^*T||^2_\alpha \leq ||T^*||^2_\alpha ||T||^2_\alpha = ||T||^2_\alpha ||T||^2_\alpha.$$

Then we have $(||T||^2_\alpha)^2 \leq ||T^*T||^2_\alpha \leq (||T||^2_\alpha)^2$. Thus
$(||T||^2_\alpha)^2 = ||T^*T||^2_\alpha$ (4.1.2)

So from (4.1.1) and (4.1.2), $||T^*T|| = (||T||)^2$.

Now replacing T by T^* we get $||T^{**}T^*|| = (||T^*||)^2$. From Theorem 3.3, $||TT^*|| = (||T||)^2$. Hence we have $||TT^*|| = ||T^*T|| = (||T||)^2$.

(7) Let $T^*T = 0$. Then $(||T||)^2 = ||T^*T|| = 0$. Thus $T = 0$.

The converse is obvious. □

Definition 4.2. A fuzzy bounded linear operator $T : H \to H$ on a Hilbert space H, is said to be self-adjoint, if $T = T^*$, where $< Tx, y >=< x, T^*y >,\ \ \forall x, y \in H$.

If T is self-adjoint operator, then $< Tx,\ y >=< x,\ Ty >$.

Theorem 4.3. *The product of two fuzzy bounded self-adjoint linear operators S and T on a fuzzy Hilbert space H is self-adjoint iff the operators commute, i.e., $ST = TS$.*

Proof. Proof is straightforward. □

Theorem 4.4. *Let $(T_n)_{n \in N}$ be a sequence of fuzzy bounded self-adjoint linear operators $T_n : H \to H$ on a fuzzy Hilbert space H. Suppose that (T_n) converges, say $T_n \to T$, i.e., $||T_n - T|| \to \bar{0}$ where $||.||$ is the fuzzy norm on the space $B(H,\ H)$. Then the limit operator T is fuzzy bounded self-adjoint linear operator on H.*

Proof. It is clear that $||T_n - T|| \to \bar{0}$. Then

$$||T_n - T||^1_\alpha \to 0 \text{ and } ||T_n - T||^2_\alpha \to 0, \quad \forall\alpha \in (0, 1].$$

Now $||T_nx - Tx|| = ||(T_n - T)x|| \preceq ||T_n - T|| \odot ||x||$
$\Rightarrow ||T_nx - Tx||^1_\alpha \leq ||T_n - T||^1_\alpha ||x||^1_\alpha$ and $||T_nx - Tx||^2_\alpha \leq ||T_n - T||^2_\alpha ||x||^2_\alpha, \ \ \forall\alpha \in (0,\ 1]$
$\Rightarrow lim_{n\to\infty}||T_nx - Tx||^1_\alpha = 0,\ lim_{n\to\infty}||T_nx - Tx||^2_\alpha = 0, \ \ \forall\alpha \in (0,\ 1]$
$\Rightarrow ||T_nx - Tx|| \to \bar{0}$
$\Rightarrow lim_{n\to\infty}T_nx = Tx.$
Since each T_n is linear $\forall n$, T is linear and T is fuzzy bounded.

Now $||T_n^* - T^*|| = ||(T_n - T)^*|| = ||T_n - T||$. From triangle inequality in $B(H,\ H)$, we have,

$$\begin{aligned} ||T - T^*|| &\preceq ||T - T_n|| \oplus ||T_n - T_n^*|| \oplus ||T_n^* - T^*|| \\ &= ||T - T_n|| \oplus \bar{0} \oplus ||T_n - T|| \\ &= \bar{2}||T_n - T|| \to \bar{0} \text{ as } n \to \infty \end{aligned}$$

Thus $||T - T^*|| = \bar{0}$ and $T = T^*$. □

5. Conclusion

In this paper, authors introduce the idea of fuzzy Hilbert adjoint operator. Existence theorem of fuzzy adjoint operator has been established.
Operator theory in fuzzy functional analysis is a recent development. So results of this paper will be helpful for the researchers in the field of operator theory in fuzzy setting.

Acknowledgements. The authors are grateful to the referees for their valuable suggestions in rewriting the paper in the present form. The authors are also thankful to the Editor-in-Chief of the journal (AFMI) for their valuable comments which helped us to revise the paper.

The present work is partially supported by Special Assistance Programme (SAP) of UGC, New Delhi, India [Grant No. F. 510/4/DRS/2009 (SAP-I)].

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